M.U. 01 H. Q.No> Give Cantor's definition of real numbers. Define addition and multiplication of real numbers.

Ans.> Cantor's definition of real numbers:-

The concept of a sequence and the limit of a sequence are basic to Cantor's theory.

Cantor developed his theory of irrational no. from the properties of convergent sequence of rational nos. The limit of a convergent sequence may be a rational no or may be an irrational number. Thus according to him, every convergent sequence of rational number defines a real no. which is represented by the sequence.

We shall start with the sequence of rational no. and we shall see that the sequence of rational no. also happens to give rise to a limit which is not rational.

Example (i). Example of a sequence of a rational no. whose limit is a rational no:

Consider the sequence $\{a_n\}$ where $a_n = \frac{n+1}{n}$.

obviously $a_n = 1 + \frac{1}{n}$

and $\frac{1}{n} \to 0$ as $n \to \alpha$.

Hence, $a_n \to 1$ as $n \to \alpha$.

Thus the sequence $\{a_n\}$ converges to the rational no. 1

Again, consider example of sequences $\{a_n\}$ given by,

$1, 1+\frac{1}{3}, 1+\frac{1}{3}+\frac{1}{3^2}, \ldots$

Here, $a_n = 1+\frac{1}{3}+\frac{1}{3^2}+ \ldots$ to n terms.

$$= \frac{1\{1-(\frac{1}{3})^n\}}{1-\frac{1}{3}}$$

$$= \frac{3}{2}\{1-(\frac{1}{3})^n\}$$

But $(\frac{1}{3})^n \to 0$ as $n \to \alpha$

That is the sequence $\{a_n\}$ converges to the limit $\frac{3}{2}$ which is a rational no.

Example (ii) :- Example of sequence of rational no. whose limit is not a rational no.

Consider the sequence,

$1, 1\cdot4, 1\cdot41, 1\cdot414, 1\cdot4142, \ldots$ which is formed by successive approximations in extracting the square root of 2.

Let the sequence be denoted by $\{a_n\}$.

We shall show that the sequence $\{a_n\}$ satisfies the cauchy condition, but it does not converge in Q.

For, $|a_{n+p} - a_n| < \frac{1}{10^{n-1}}$

which can be made as small as we ~~plae~~ please by taken n sufficiently large.

Hence the sequence $\{a_n\}$ satisfies the

Cauchy's Condition.

Again, in the given sequence,

$$a_n^2 < 2 < \left(a_n + \frac{1}{10^{n-1}}\right)^2$$

$$\Rightarrow 0 < 2 - a_n^2 < \left(a_n + \frac{1}{10^{n-1}}\right)^2 - a_n^2$$

$$= \frac{2a_n}{10^{n-1}} + \frac{1}{10^{2(n-1)}}$$

$$\Rightarrow 0 < 2 - a_n^2 < \frac{3}{10^{n-1}} + \frac{1}{10^{2(n-1)}},$$

Since $a_n < 1.5$

$< \varepsilon$, by taking n sufficiently large

$$\therefore \text{Lt } a_n^2 = 2$$

$$\Rightarrow \text{Lt } a_n = \sqrt{2}$$

But $\sqrt{2}$ is not a rational no. Thus we see the sequence of rational no. may also have a limit which is not rational.

Addition of real nos. :—

The sum $A+B$ of two real nos. represented by $\{a_n\}$ and $\{b_n\}$ is defined to be the real no. represented by the sequence $\{a_n + b_n\}$ and the difference $A-B$ is defined as a no. represented by $\{a_n - b_n\}$.

Multiplication of real nos. :—

The Product AB, of two real nos. is defined to be the no. represented by the sequence $\{a_n b_n\}$.

The quotient $\frac{A}{B}$ is defined to be the no. represented by the convergent sequence $\left\{\frac{a_n}{b_n}\right\}$ except when B is zero.

on the basis of above definitions the laws governing the addition and multiplication of rational nos. may be extended to the class of real nos.

QNo→ Define Convergent and divergent sequence.

Ans.→ Convergent sequence :- A sequence which converges to some no. l is said to be convergent sequence.

Example:- (i) A constant sequence $(a, a, a, \text{-------})$ of real no converges to the real no. a

(ii) The sequence (a_n) where, $a_n = \frac{1}{n}$ converges to to the real no. (zero).

Divergent sequence:- A sequence is divergent if it is not convergent i.e. if it does not have a limit.

Example:- (i) The sequence $\{a_n\}$ where, $a_n = n (n = 1, 2, 3 \text{-----})$ is divergent

(ii) The sequence $\{a_n\}$, where $a_n = \log\left(\frac{1}{n}\right)$ diverges to $-\infty$

QNo→ Define Cauchy's sequence

Ans→ Cauchy's sequence:-

A sequence $\{a_n\}$ is said to be a cauchy's sequence if for every $\epsilon > 0$ there exists $M \in N$ such that.

$|a_m - a_n| < \epsilon \ (m, n \geq M)$.